Symmetries of the Khokhlov-Zabolotskaya equation

1987 J. Phys. A: Math. Gen. 20 1613

(http://iopscience.iop.org/0305-4470/20/6/041)

J. Phys. A: Math. Gen. 20 (1987) 1613-1614. Printed in the UK

COMMENT

Symmetries of the Khokhlov–Zabolotskaya equation

F Schwarz

Gesellschaft für Mathematik und Datenverarbeitung, Institut F1, Postfach 1240, 5205 Sankt Augustin, West Germany

Received 11 August 1986

Abstract. The group of Lie symmetries of the Khokhlov-Zabolotskaya equation in two and three space dimensions is given.

There has recently appeared in this journal an article by Chowdhury and Nasker (1986) claiming to derive the symmetry group of the Khokhlov-Zabolotskaya equations which describes the propagation of sound in a non-linear medium. The results given there are incomplete and partially wrong. It is the purpose of this comment to report the complete answer for both the two- and the three-dimensional cases. By a suitable choice of constants the equations under consideration may be written as

$$u_{tx} - (uu_x)_x - u_{yy} = 0 \tag{1}$$

or

$$u_{tx} - (uu_x)_x - u_{yy} - u_{zz} = 0 \tag{2}$$

in two or three space dimensions respectively (Rudenko and Soluyan 1977, p 215). To obtain the full group of Lie symmetries of these equations the REDUCE package SPDE which has been developed by Schwarz (1985a, b, 1986) is applied. Upon submission of the equations (1) or (2) it determines the full symmetry group completely automatically. The results may be described as follows. In the two-dimensional case (1) there is one infinitesimal generator which generates a finite subgroup. In addition there are three generators depending on three arbitrary functions of time, i.e. they correspond to an infinite-dimensional Lie algebra. Explicitly the generators are

$$U_1 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u}$$

$$U_2 = g(t)\frac{\partial}{\partial x} - g^1(t)\frac{\partial}{\partial u}$$

$$U_3 = \tfrac{1}{2}yh'(t)\frac{\partial}{\partial x} + h(t)\frac{\partial}{\partial y} - \tfrac{1}{2}yh''(t)\frac{\partial}{\partial u}$$

$$U_4 = \left[f(t)\frac{\partial}{\partial t} + \tfrac{1}{3}xf'(t) + \tfrac{1}{6}y^2f''(t)\right]\frac{\partial}{\partial x} + \tfrac{2}{3}yf'(t)\frac{\partial}{\partial y} - [\tfrac{2}{3}uf'(t) + \tfrac{1}{3}xf''(t) + \tfrac{1}{6}y^2f'''(t)]\frac{\partial}{\partial u}.$$

The generator U_1 corresponds to X_1 of Chowdhury and Nasker if a missing y is added to the last term. If $f(t)=t$ is substituted into U_4, the generator X_3 of these authors is obtained if a missing ρ is added to the first term.

In the three-dimensional case which is described by (2) the full Lie algebra of point symmetries is also infinite depending again on three arbitrary functions of time. They may be written as

$$U_1=\frac{\partial}{\partial t} \qquad U_2=z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z}$$

$$U_3=2x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+2u\frac{\partial}{\partial u}$$

$$U_4=5t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+3y\frac{\partial}{\partial y}+3z\frac{\partial}{\partial z}-4u\frac{\partial}{\partial u}$$

$$U_5=10t^2\frac{\partial}{\partial t}+(4tx+3y^2+3z^2)\frac{\partial}{\partial x}+12ty\frac{\partial}{\partial y}+12tz\frac{\partial}{\partial z}-4(4tu+x)\frac{\partial}{\partial u}$$

$$U_6=f(t)\frac{\partial}{\partial x}-f'(t)\frac{\partial}{\partial u}$$

$$U_7=\tfrac{1}{2}yg'(t)\frac{\partial}{\partial x}+g(t)\frac{\partial}{\partial z}-\tfrac{1}{2}yg''(t)\frac{\partial}{\partial u}$$

$$U_8=\tfrac{1}{2}zh'(t)\frac{\partial}{\partial x}+h(t)\frac{\partial}{\partial y}-\tfrac{1}{2}zh''(t)\frac{\partial}{\partial u}.$$

By choosing a special ansatz for the functions $f(t)$, $g(t)$ and $h(t)$ often a finite subgroup is generated which leads to a simple reduction and corresponding similarity solutions. The meaning of these infinite symmetry groups with respect to possible conservation laws does not seem to be obvious.

References

Chowdhury A R and Nasker M 1986 *J. Phys. A: Math. Gen.* **19** 1175

Rudenko O V and Soluyan S I 1977 *Theoretical Foundations of Nonlinear Acoustics* (New York: Consultants Bureau)

Schwarz F 1985a *Computing* **34** 91

—— 1985b *Computing* **36** 279

—— 1986 *The Package SPDE for Determining Symmetries of Partial Differential Equations, User's Manual* to be distributed with REDUCE 3.3